JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **35**, No. 1, February 2022
http://dx.doi.org/10.14403/jcms.2022.35.1.91

THE SPECTRAL DECOMPOSITION FOR FLOWS ON TVS-CONE METRIC SPACES

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Abstract. We study some properties of nonwandering set $\Omega(\phi)$ and chain recurrent set $\mathrm{CR}(\phi)$ for an expansive flow which has the POTP on a compact TVS-cone metric spaces. Moreover we shall prove a spectral decomposition theorem for an expansive flow which has the POTP on TVS-cone metric spaces.

1. Introduction and preliminaries

In 1967, Smale had proved the spectral decomposition theorem, i.e, the nonwandering set of an Axiom A dynamical system on a compact manifold is the union of finitely many basic sets[4]. Also, the spectral decomposition theorem for an expansive flow with the pseudo orbit tracing property on a compact uniform space had proved by J. S. Park and S. H. Ku in 2020[2].

In this paper we investigate some properties of nonwandering set and chain recurrent set for flows on a compact TVS-cone metric space.

And we extend the spectral decomposition theorem on a compact uniform space to expansive flow with the pseudo orbit tracing property on a compact TVS-cone metric space.

We now introduce notions and definitions necessary for our works,

Let E be a topological vector space. A subset P of E is called a topological vector space cone (abbr. TVS-cone) if the following are satisfied

(A) P is closed and $\mathrm{Int}(P) \neq \emptyset$

Received December 29, 2021; Accepted February 16, 2022.

2010 Mathematics Subject Classification: Primary 37B40; Secondary 37B02.

Key words and phrases: TVS-cone metric space, flow, Nonwandering set and chain recurrent set, POTP, expansiveness, stable and unstable sets, topologically transitive, The spectral decomposition theorem.

(B) If $u, v \in P$ and $a, b \geq 0$, then $au + bv \in P$

(C) If $u, -u \in P$, then $u = 0$.

Let P be a TVS-cone of a topological vector space E. Some partial ordering $\leq$, $<$ and $\ll$ on E with respect to P are defined as followings respectively

(i) $u \leq v$ if $v - u \in P$

(ii) $u < v$ if $u \leq v$ but $u \neq v$

(iii) $u \ll v$ if $v - u \in \mathrm{Int}(P)$, where $\mathrm{Int}(P)$ denote the interior of P.

Lemma 1.1. *Let P be a TVS-cone of a topological vector space E. Then the following hold.*

(1) If $u \gg 0$, then $ru \gg 0$ for all $r > 0$

(2) If $u_1 \gg v_1$, $u_2 \gg v_2$, then $u_1 + u_2 \gg v_1 + v_2$

(3) If $u \gg 0$ and $v \gg 0$, then there exists $w \gg 0$ such that $w \ll u$ and $w \ll v$ [1].

Let E be a topological vector space with cone P. A map $d : X \times X \to E$ is called a TVS-cone metric on X and (X, d) called a TVS-cone metric space if the following conditions are satisfied.

(i) $d(x, y) \geq 0$ for all $(x, y) \in X \times X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $(x, y) \in X \times X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z in X.

Let (X, d) be a TVS-cone metric space, then the collection of all u-balls $B_d(x, u)$, $\mathfrak{B} = \{B_d(x, u) \mid x \in X, u \gg 0\}$, is a basis for some topology $\Im$ on X.

In this paper, we always suppose that a cone P is a TVS-cone on a topological vector space E and a TVS-cone metric space (X, d) is a topological space with the above topology $\Im$.

Let (X, d) be a TVS-cone metric space over topological vector space E.

A flow on X is the triplet $(X, \mathbb{R}, \phi$, where ϕ is a map from the product space $X \times \mathbb{R}$ into the space X satisfying the following axioms :

(1) $\phi(x, 0) = x$ for every $x \in X$,

(2) $\phi(\phi(x, s), t) = \phi(x, s + t)$ for every $x \in X$ and s, t in $\mathbb{R}$,

(3) ϕ is continous.

We denoted by $C_0(\mathbb{R})$ the set of all continuous functions $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$.

Let ϕ be a flow on a TVS-cone metric space (X, d) :

ϕ is said to be expansive if for every $\epsilon > 0$ there exists a vector $u \gg 0$ such that if $x, y \in X$ satisfy $d(\phi_t(x), \phi_{h(t)}(y)) \ll u$ for all $t \in \mathbb{R}$ and some $h \in C_0(\mathbb{R})$ then $y = \phi_r(x)$ where $|r| < \epsilon$.

Let ϕ be a flow on a TVS-cone metric space (X, d). Given a vector $u \gg 0$ and a real number $T > 0$, an (u, T)-pseudo orbit is a collection of sequences $(\{x_i\}, \{t_i\})$ so that $t_i \geq T$ and $d(\phi_{t_i}(x_i), x_{i+1}) \ll u$ for all $i \in \mathbb{Z}$. For the sequence $\{t_i\}$ we write $s_n = \sum_{i=0}^{n-1} t_i, s_{-n} = \sum_{i=-n}^{-1} t_i$, where $s_0 = \sum_{i=0}^{-1} t_i = 0$. we always assume $\sum_{i=j}^{k} t_i = 0$ if $k < j$.

An (u, T)-pseudo orbit $(\{x_i\}, \{t_i\})$ is v-traced by an orbit $(\phi_t(x))_{t\in\mathbb{R}}$ if

$$d(\phi_t(z), \phi_{t-s_n}(x_n)) \ll v \text{ if } s_n \leq t < s_{n+1} \text{ for } n \geq 0$$

and

$$d(\phi_t(z), \phi_{t+s_n}(x_n)) \ll v \text{ if } -s_n \leq t < -s_{n+1} \text{ for } n < 0.$$

We say that a flow ϕ has the pseudo orbit tracing property (POTP) if for any vector $u \gg 0$ there exists a vector $v \gg 0$ such that every (v, T)-pseudo orbit is u-traced by an orbit of ϕ for all $T > 0$.

Lemma 1.2. *Let ϕ be a flow on a TVS-cone metric space (X, d) without fixed points. Then there exists a number $T_0 > 0$ such that for every $T \in [0, T_0]$ there is a vector $u \gg 0$ such that $d(\phi_T(x), x) \geq u$ for all $x \in X$* [1] .

Lemma 1.3. [1] *Let ϕ be a flow on a TVS-cone metric space (X, d). Let $\mathcal{T}$ be a compact subset of $\mathbb{R}$ and $x \in X$. Then for every vector $u \gg 0$ there exists a vector $v \gg 0$ such that if $d(x, y) \ll v$, then $d(\phi_t(x), \phi_t(y)) \ll u$ for all $t \in \mathcal{T}$.*

Lemma 1.4. *Let ϕ be a flow on a TVS-cone metric space (X, d) without fixed points and let T_0 be the number determined by Lemma 1.3. For every $T \in (0, T_0)$ there exists a vector $u \gg 0$ with $d(\phi_T(x), y) \geq u$ provided that $d(x, y) \ll u$.*

Proof. Take a vector $v \gg 0$ determined in Lemma 1.2 and choose a vector w_1, with $0 \ll w_1 \ll \frac{1}{2}v$. Since ϕ_T is uniformly continuous, there exists a vector $w_2 \gg 0$ such that if $d(x, y) \ll w_2$, then $d(\phi_T(x), \phi_T(y)) \ll w_1$. Moreover, if $d(x, y) \ll w_2$, then $d(\phi_T(x), y) \gg w_1$. Assume that $d(\phi_T(x), y) \ll w_1$.

We obtain $d(\phi_T(y), y) \leq d(\phi_T(y), \phi_T(x)) + d(\phi_T(x), y) \ll 2w_1 \ll v$, contradicting that the choice of v. Take a vector $u \gg 0$ with $u \ll w_1$, and $u \ll w_2$. If $d(x, y) \ll u \ll w_2$, then $d(\phi_T(x), y) \gg w_1$, so that $d(\phi_T(x), y) \gg u$. □

2. Nonwandering set and chain recurrent set

Let ϕ be a flow on a TVS-cone metric space (X,d). Given a vector $u \gg 0, T > 0$, and $x, y \in X$, an (u,T)-chain from x to y is a collection $\{x = x_0, x_1, \cdots, x_{n-1}, x_n = y; t_0, t_1, \cdots, t_{n-1}\}$ so that $t_i \geq T$ and $d(\phi_{t_i}(x_i), x_{i+1}) \ll u$ for all $i = 0, 1, \cdots, n-1$.

A point x is equivalent to y, written $x \sim y$, if for every vector $u \gg 0$ and $T > 0$, there is an (u,T)-chain from x to y and one from y to x. The chain recurrent set of ϕ is $\mathrm{CR}(\phi) = \{x \in X | x \sim x\}$.

The relation $\sim$ is an equivalence relation on $\mathrm{CR}(\phi)$ and the equivalence classes are called chain component for ϕ.

A point $x \in X$ is called nonwandering of a flow ϕ on a TVS-cone metric space (X,d) if for any neighborhood U of x, $\phi_T(U) \cap U \neq \emptyset$ for some $T > 0$. The set of nonwandering points is denoted by $\Omega(\phi)$.

Lemma 2.1. *Let ϕ be a flow on a compact TVS-cone metric space (X,d). If $x \in \mathrm{CR}(\phi)$, then $x \sim \phi_r(x)$ for all $r \in \mathbb{R}$.*

Proof. Let $r > 0, T > 0$ and a vector $u \gg 0$. If $T \leq r$, then $\{x, \phi_r(x); r\}$ is an (u,T)-chain from x to $\phi_r(x)$. Now consider the case $T > r$. By the continuity of ϕ_r, there exists a vector $v \gg 0$ such that if $d(x,y) \ll v$, then $d(\phi_r(x), \phi_r(y)) \ll u$. By $x \sim x$, there is a (v,T)-chain $\{x = x_0, \cdots, x_k = x; t_0, t_1, \cdots, t_{k-1}\}$ from x to itself. Then a collection

$$\{x = x_0, \cdots, x_{k-1}, \phi_r(x); t_0, t_1, \cdots, t_{k-1} + t\}$$

is an (u,T)-chain from x to $\phi_r(x)$.

Also, since there exists an $(u, T+r)$-chain

$$\{x = x_0, \cdots, x_k = x; t_0, t_1, \cdots, t_{k-1}\}$$

from x to itself, we get an (u,T)-chain from $\phi_r(x)$ to x

$$\{\phi_r(x), x_1, \cdots, x_k = x; t_0 - r, t_1, \cdots, t_{k-1}\}.$$

Therefore $x \sim \phi_r(x)$. For $r < 0$, it follows that $x \sim \phi_r(x)$ by similar argument. $\square$

A set $M \subset X$ is said to be invariant if $\phi_t(M) \subset M$ for all $t \in \mathbb{R}$.

Lemma 2.2. *Let ϕ be a flow on a compact TVS-cone metric space (X,d) and C be a chain component. Then C is invariant and closed.*

Proof. By Lemma 2.1, C is invariant. To show the closedness of C, we shall prove that $C = \overline{C}$. Let $z \in \overline{C}$. To prove that $z \in C$, let $y \in C$ and let $T > 0$ and a vector $u \gg 0$. By the continuity of ϕ_T, there exists a vector v with $0 \ll v \ll \frac{1}{2}u$ such that if $d(x,y) \ll v$, then $d(\phi_T(x), \phi_T(y)) \ll u$. From $z \in \overline{C}$, there is a $x \in C$ such that $d(x, z) \ll v$. Because $x, y \in C$, there exists a $(v, 2T)$-chain

$$\{x = x_0, x_1, \cdots, x_k = y; t_0, t_1, \cdots, t_{k-1}\}$$

from x to y. Since $d(x, z) \ll v$, we have $d(\phi_T(x), \phi_T(z)) \ll u$. so a collection

$$\{z, \phi_T(x), x_1, \cdots, x_k = y; T, t_0 - T, t_1, \cdots, t_{k-1}\}$$

is an (u, T)-chain from x to y.

Now we shall obtain an (u, T)-chain from y to z. Notice that there exists a (v, T)-chain

$$\{y = x_0, x_1, \cdots, x_k = y; t_0, \cdots, t_{k-1}\}$$

from y to x. Then

$$d(\phi_{t_{k-1}}(x_{k-1}), z) \leq d(\phi_{t_{k-1}}(x_{k-1}), x) + d(x, z) \ll 2v \ll u.$$

Thus a collection

$$\{y = x_0, \cdots, x_{k-1}, z; t_0, \cdots, t_{k-1}\}$$

is an (u, T)-chain from y to z.

Concatenating the (u, T)-chain from z to y with the (u, T)-chain from y to z, we obtain an (u, T)-chain from z to itself. Therefore $z \in \mathrm{CR}(\phi)$ and $z \sim y$. Consequently $z \in C$, i.e., C is closed. □

PROPOSITION 2.3. *Let ϕ be a flow on a compact TVS-cone metric space (X, d). If ϕ has the POTP, then $\Omega(\phi) = \mathrm{CR}(\phi)$.*

Proof. Let $x \in \mathrm{CR}(\phi)$. Given any neighborhood U of x, there exists a vector $u \gg 0$ such that $B(x, u) \subset U$. Since ϕ has the POTP, there is a vector $v \gg 0$ such that every $(v, 1)$-pseudo orbit is u-traced by some orbit. By $x \in \mathrm{CR}(\phi)$, there exists a $(v, 1)$-chain

$$\{x = x_0, x_1, \cdots, x_{k-1}, x_k = x; t_0, t_1, \cdots, t_{k-1}\}$$

from x to itself.

For $n = mk + j$ with $m \in \mathbb{Z}$ and $0 \leq j < k$, let $x_n = x_j$ and $t_n = t_i$. Then $(\{x_n\}, \{t_n\})$ is a $(v, 1)$-pseudo orbit.

Thus there exists $z \in X$ such that

$$d(\phi_t(z), \phi_{t-s_n}(x_n)) \ll u \text{ for } s_n \leq t < s_{n+1}, n \geq 0$$

and

$$d(\phi_t(z), \phi_{t-s_n}(x_n)) \ll u \text{ for } -s_n \leq t < -s_{n+1}, n \leq 0.$$

From the fact that $d(z, x) = d(\phi_0(z), \phi_{0-s_0}(x)) \ll u$, we have $z \in B(x, u) \subset U$. There exists an $n = mk$ such that $s_n > 1$.

By $d(\phi_{s_n}(z), x) = d(\phi_{s_n}(z), \phi_{s_n - s_n}(x_n)) \ll u$, we have $\phi_{s_n}(z) \in B(x, u) \subset U$, i.e., $\phi_{s_n}(U) \cap U \neq \emptyset$. Consequently, $x \in \Omega(\phi)$. Since $\Omega(\phi) \subset \mathrm{CR}(\phi)$, we conclude that $\Omega(\phi) = \mathrm{CR}(\phi)$. □

PROPOSITION 2.4. *Let ϕ be an expansive flow on a compact TVS-cone metric space (X, d). If ϕ has the POTP, then the set* $\mathrm{Per}(\phi)$ *of periodic points is dense in* $\mathrm{CR}(\phi)$.

Proof. Since $\mathrm{Per}(\phi) \subset \mathrm{CR}(\phi)$ and $\mathrm{CR}(\phi)$ is closed, we get $\overline{\mathrm{Per}(\phi)} \subset \mathrm{CR}(\phi)$.

To prove that $\mathrm{CR}(\phi) \subset \overline{\mathrm{Per}(\phi)}$, choose $x \in \mathrm{CR}(\phi)$. Let U be a neighborhood of x and let $0 < \epsilon < 1$. we claim that $U \cap \mathrm{Per}(f) \neq \emptyset$.

By the expansiveness, there exists a vector $u \gg 0$ with $B(x, u) \subset U$ such that if

$$d(\phi_{f(t)}(x), \phi_t(y)) \ll u \text{ for all } t \in \mathbb{R} \text{ and some } f \in C_0(\mathbb{R})$$

then $y = \phi_s(x)$ for some $|s| < \epsilon$.

Since ϕ has the POTP, there is a vector $v \gg 0$ such that every $(v, 1)$-pseudo orbit is $\frac{1}{2}u$-traced by some orbit of ϕ.

Since $x \in \mathrm{CR}(\phi)$, there exists a $(v, 1)$-chain $\{x = x_0, x_1, \cdots, x_{k-1}, x_k = x; t_0, t_1, \cdots, t_{k-1}\}$ from x to itself.

We can extend this $(v, 1)$-chain to a $(v, 1)$-pseudo orbit $(\{x_n\}, \{t_n\})$ in a same way as the proof of Proposition 2.3. Then there exists $z \in X$ such that

$$d(\phi_t(z), \phi_{t-s_n}(x_n)) \ll \frac{1}{2}u \text{ for } s_n \leq t < s_{n+1}, n \geq 0$$

and

$$d(\phi_t(z), \phi_{t-s_n}(x_n)) \ll \frac{1}{2}u \text{ for } -s_n \leq t < -s_{n+1}, n \leq 0.$$

Let $m \geq 0$, $s_{mk+j} \leq t < s_{mk+j+1}$. Since

$$s_{(m+1)k+j} = s_{mk+j} + s_k \leq t + s_k < s_{mk+j+1} + s_k = s_{(m+1)k+j+1},$$

we have

$$d(\phi_{t+s_k}(z), \phi_{t+s_k - s_{(m+1)k+1}}(x_{(m+1)k+j})) = d(\phi_{t+s_k}(z), \phi_{t-s_{mk+j}}(x_{mk+j})) < \frac{1}{2}u.$$

Thus $d(\phi_{t+s_k}(z), \phi_t(z)) \leq d(\phi_{t+s_k}(z), \phi_{t-s_n}(x_n)) + d(\phi_{t-s_n}(x_n), \phi_t(z)) \ll u$.

Let $m < 0, -s_{mk+j} \leq y < -s_{mk+j+1}$. Since

$$-s_{(m-1)k+j} = -s_{mk+j} + s_k \leq t + s_k \leftarrow s_{mk+j+1} + s_k = -s_{(m-1)k+j+1},$$

we have

$$d(\phi_{t+s_k}(z), \phi_{t+s_k+s_{(m-1)k+j}}(x_{(m-1)k+j})) = d(\phi_{t+s_k}(z), \phi_{t+s_{mk+j}}(x_{mk+j})) < \frac{1}{2}u.$$

Thus $d(\phi_{t+s_k}(z), \phi_t(z)) \le d(\phi_{t+s_k}(z), \phi_{t+s_n}(x_n)) + d(\phi_{t+s_n}(x_n), \phi_t(z)) \ll u$. From $d(\phi_{t+s_k}(z), \phi_t(z)) = d(\phi_t(\phi_{s_k}(z)), \phi_t(z)) \ll u$ for all $t \in \mathbb{R}$, we have

$$z = \phi_s(\phi_{s_k}(x)) = \phi_{s_k+s}(z) \text{ for some } |s| < \epsilon.$$

By $s_k \ge s_1 = t_0 \ge 1 > \epsilon > |s|$, we get $s_k - s \ge s_k - |s| > 0$. Thus z is a periodic point. Moreover $d(z,x) = d(\phi_0(z), \phi_{0-s_0}(x_0)) \ll \frac{1}{2}u \ll u$ and therefore $z \in B(x,u) \subset U$.

Consequently the set $\mathrm{Per}(\phi)$ of periodic points is dense in $\mathrm{CR}(\phi)$. □

Remark 2.5. Assume that a flow is expansive and has the POTP. In the proof of Proposition 2.4, we demonstrate that if a pseudo orbit is periodic, then there exists a periodic point which traces the periodic pseudo orbit.

3. Spectral decomposition theorem

Smale's spectral decomposition theorem say that for Axiom A flows the nonwandering set partitions into nonempty closed invariant sets each of which is topologically transitive [4].

We now prove spectral decomposition theorem for an expansive flow on a compact TVS-cone metric space. First we introduce the following definitions and lemmas.

Let ϕ be a flow on a TVS-cone metric space (X,d). For given vector $u \gg 0$ and $x \in X$, let $W^s_u(x)$ and $W^u_u(x)$ be the local stable and local unstable sets defined by

$$W^s_u(x) = \{y \in X \mid d(\phi_t(x), \phi_t(y)) \ll u \text{ for all } t \ge 0\}$$

$$W^u_u(x) = \{y \in X \mid d(\phi_t(x), \phi_t(y)) \ll u \text{ for all } t \le 0\}.$$

Also, define the stable and unstable sets $W^s(x), W^u(x)$ as

$$W^s(x) = \{y \in X \mid \forall u \gg 0, \exists T > 0 \text{ s.t. } \mathcal{O}(x) \cap B(\phi_t(y), u) \ne \emptyset \text{ for all } t \ge T\}$$

$$W^u(x) = \{y \in X \mid \forall u \gg 0, \exists T < 0 \; s.t. \; \mathcal{O}(x) \cap B(\phi_t(y), u) \ne \emptyset \text{ for all } t \le T\}$$

, where $\mathcal{O}(x)$ denote the orbit of x.

Lemma 3.1. *Let ϕ be an expansive flow on a compact TVS-cone metric space (X,d). Then there exists a vector $u \gg 0$ such that if $p \in \mathrm{Per}(\phi)$, then $W^s_u(p) \subset W^s(p)$ and $W^u_u(p) \subset W^u(p)$.*

Proof. Let $u_1 \gg 0$ be an expansive vector with respect to 1. Choose any vector u, $0 \ll u \ll u_1$. We claim that $W^s_u(p) \subset W^s(p)$.

Assume that $W^s_u(p) \not\subset W^s(p)$ for some periodic point p. We take $y \in W^s_u(p) - W^s(p)$. By $y \notin W^x(p)$, there is a vector $w \gg 0$ such that for every $t > 0$,

$$\mathcal{O}(p) \cap B(\phi_T(y), w) = \emptyset \text{ for some } T \geq t.$$

So for each n there exists a $t_n > \max\{n, t_{n-1}\}$ such that $\mathcal{O}(p) \cap B(\phi_{t_n}(y), w) = \emptyset$. By the compactness of X, the sequence $\{\phi_{t_n}(p)\}$ has a convergent subsequence.

Let $\phi_{t_n}(y) \to y_0$. We claim that $\mathcal{O}(p) \cap B(y_0, \frac{1}{2}w) = \emptyset$. Otherwise, there exists a $q \in \mathcal{O}(p) \cap B(y_0, \frac{1}{2}w)$. By $\phi_{t_n}(y) \to y_0$, there is a k such that $\phi_{t_k}(y) \in B(y_0, \frac{1}{2}w)$. Since $d(q, \phi_{t_k}(y)) \leq d(q, y_0) + d(y_0, \phi_{t_k}(q)) \ll \frac{1}{2}w + \frac{1}{2}w = w$, it follows that $q \in \mathcal{O}(p) \cap B(\phi_{y_k}(y), w)$. This is contradiction. Thus $y_0 \notin \mathcal{O}(p)$.

Let $\phi_{t_n}(p) \to p_0$. For any $t \in \mathbb{R}$, since $t_n \to \infty$, there is a positive integer N such that $t_N + t > 0$. By $t_n + t \geq t_N + t$ for all $n \geq N$, $d(\phi_t \phi_{t_n}(p), \phi_t \phi_{t_n}(y)) \ll u$. Put $n \to \infty$. Then $d(\phi_t(p_0), \phi_t(y_0)) \ll u \ll u_1$. Thus $y_0 = \phi_s(p_0)$ for some s, $|s| < 1$ and hence $y_0 \in \mathcal{O}(p)$.

This contradiction imply that $W^s_u(p) \subset W^s(p)$ for all $p \in \mathrm{Per}(\phi)$. The proof for $W^u_u(p) \subset W^u(p)$ is similar. □

LEMMA 3.2. *Let $p, q \in \mathrm{Per}(\phi)$. If $W^u(p) \cap W^s(q) \neq \emptyset$ and $W^s(p) \cap W^u(q) \neq \emptyset$, Then $p \sim q$.*

Proof. Let $x \in W^u(p) \cap W^s(q)$. Let any vector $u \gg 0$ and $T > 0$. Then there is a $s > 0$ such that $\mathcal{O}(p) \cap B(\phi_t(x), u) \neq \emptyset$ for all $t \leq -s$ and $\mathcal{O}(q) \cap B(\phi_t(x), u) \neq \emptyset$ for all $t \geq s$. choose $t \geq s$ with $2t \geq T$.

Let $p_0 \in \mathcal{O}(p) \cap B(\phi_{-t}(x), u)$ and $q_0 \in \mathcal{O}(q) \cap B(\phi_t(x), u)$. Take $r_1, r_2 \geq T$ such that $p_0 = \phi_{r_1}(p)$ and $q_0 = \phi_{r_2}(q)$, respectively.

Then $\{p, \phi_{-t}(x), q_0, q; r_1, 2t, r_2\}$ is an (u, T)-chain form p to q.

Similarly, we can construct an (u, T)-chain from q to p. Consequently, $p \sim q$. □

LEMMA 3.3. *Let ϕ be an expansive flow on a compact TVS-cone metric space (X, d). If ϕ has the POTP and C is a chain component, then C is open in $\Omega(\phi)$.*

Proof. Let $u_1 \gg 0$ be the vector determined as Lemma 3.1. Take a vector $v_1 \gg 0$ corresponding to u_1 by the POTP. By Proposition 2.3 and 2.4, $\Omega(\phi) = \overline{\mathrm{Per}(\phi)}$. Since $U \equiv B(C, \frac{1}{2}v_1) \cap \Omega(\phi)$ is a nonempty open set in $\Omega(\phi)$, $U \cap \mathrm{Per}(\phi)$ is nonempty. Let $p \in U \cap \mathrm{Per}(\phi)$. Then $d(y, p) \ll \frac{1}{2}v_1$ for some $y \in C$.

We claim that $p \sim y$. For any vector $u \gg 0$ and number $T > 0$, there exists a vector $v \gg 0$ with $v \ll \frac{1}{2}v_1, v \ll u$ such that if $d(y,z) \ll v$, then $d(\phi_T(y), \phi_T(z)) \ll u$. We can take $q \in B(y,v) \cap \mathrm{Per}(\phi)$ by $B(y,v) \cap \mathrm{Per}(\phi) \neq \emptyset$. $d(y,q) \ll v$ implies $d(\phi_T(y), \phi_T(q)) \ll u$. By the periodicity of q, $\phi_t(q) = q$ for some t, $t \geq 2T$. Then $\{y, \phi_T(q), q; T, t-T\}$ is an (u,T)-chain from y to q and $\{q, y; t\}$ is an (u,T)-chain from q to y.

Take $r_1, r_2 \geq T$ with $\phi_{r_1}(p) = p$ and $\phi_{r_2}(q) = q$. we define the following

$$u_n = \phi_{-nr_1}(p),\ t_n = r_1 \text{ for } n < 0$$
$$u_n = \phi_{nr_2}(q),\ t_n = r_2 \text{ for } n \geq 0.$$

Then $d(p,q) \leq d(p,y) + d(y,q) \ll \frac{1}{2}v_1 + v \ll \frac{1}{2}v_1 + \frac{1}{2}v_1 = v_1$. Therefore $(\{v_n\}, \{t_n\})$ is a (v_1,T)-pseudo orbit. Thus there exists a $z \in X$ that u_1-traces the (v_1,T)-pseudo orbit $(\{u_n\}, \{t_n\})$. For $t \geq 0$, let $nr_2 \leq t < (n+1)r_2$. Then $d(\phi_t(z), \phi_{t-nr_2}(x_n)) = d(\phi_t(z), \phi_{t-nr_2}(q)) = d(\phi_t(z), \phi_t(q)) \leq u_1$. Thus $z \in W^s_{u_1}(q)$. For $t \leq 0$, let $-nr_1 \leq t < -(n-1)r_1$.

Then $d(\phi_t(z), \phi_{t+nr_1}(x_n)) = d(\phi_t(z), \phi_{t+nr_1}(\phi_{-nr_1})(p)) = d(\phi_t(z), \phi_t(p)) \ll u_1$. Thus $z \in W^u_{u_1}(p)$ and it follows that $z \in W^u_{u_1}(p) \cap W^s_{u_1}(q) \subset W^u(p) \cap W^s(q)$.

Define

$$v_n = \phi_{-nr_2}(q),\ t_n = r_2 \text{ for } n < 0$$
$$v_n = \phi_{nr_1}(p),\ t_n = r_1 \text{ for } n \geq 0,$$

then $(\{v_n\}, \{t_n\})$ is also a (v_1,T)-pseudo orbit. Thus there exists a $w \in X$ that u_1-traces the (v_1,T)-pseudo orbit $(\{v_n\}, \{t_n\})$. Then $z \in W^s_{u_1}(p) \cap W^u_{u_1}(q) \subset W^s(p) \cap W^u(q)$. By Lemma 3.2, $p \sim q$ and there exists an (u,T)-chain from p to q and one from q to p.

Concatenating the (u,T)-chain from p to q with the (u,T)-chain from q to y, we obtain an (u,T)-chain form p to y. In the similar process we find an (u,T)-chain from y to p. Therefore $p \sim y$. Thus $p \in C$ that is $U \cap \mathrm{Per}(\phi) \subset C$. Since C is closed in X, we have $C \supset \overline{U \cap \mathrm{Per}(\phi)} \supset U \cap \overline{\mathrm{Per}(\phi)} = U$. By $C \subset U$, and so we conclude that $C = U$. Thus it follows that C is open in $\Omega(\phi)$. □

Let ϕ be a flow on a TVS-cone compact metric space (X,d). we recall that a set $A \subset X$ is said to be topologically transitive with respect to ϕ if for any open sets U and V in X such that $U \cap A$ and $V \cap A$ are nonempty, there exists $T > 0$ such that $\phi_T(U) \cap V \neq \emptyset$.

Proposition 3.4. *Let ϕ be a flow on a compact TVS-cone metric space (X,d). If ϕ has the POTP and C is a chain component, then C is topologically transitive with respect to the flow ϕ.*

Proof. Let U, V be nonempty open sets in C and let $x \in U, y \in V$. Then there exists a vector $u \gg 0$ such that $B(x,u) \cap C \subset U, B(y,u) \cap C \subset V$ and $B(x,u) \cap \mathrm{CR}(\phi) \subset C, B(y,u) \cap \mathrm{CR}(\phi) \subset C$. Take a vector $v \gg 0$ satisfying definition of the POTP with respect to u.

By $x \sim y$, there is a $(v,1)$-chain from x to itself passing y $\{x_0 = z, \cdots, x_n = y, \cdots, x_m = x; t_0, \cdots, t_{m-1}\}$ Then we can extend $(v,1)$-chain to a periodic $(v,1)$-pseudo orbit $(\{x_i\}, \{t_i\})$. By Remark 2.5, there is a periodic point $z \in X$ that u-traces the periodic $(v,1)$-pseudo orbit $(\{x_i\}, \{t_i\})$.

Since $d(z,x) = d(\phi_0(z), \phi_{0-s_0}(x_0)) \ll u$, we have $z \in B(x,u) \cap \mathrm{CR}(\phi) \subset B(x,u) \cap C \subset U$. By $d(\phi_{s_n}(z), y) = d(\phi_{s_n}(z), \phi_{s_n - s_n}(x_n)) \ll u$, we obtain $\phi_{s_n}(z) \in B(y,u) \cap \mathrm{CR}(\phi) \subset B(y,u) \cap C \subset V$. Consequently, $\phi_{s_n}(U) \cap V \neq \emptyset$.

Therefore C is topologically transitive with respect to ϕ. □

THEOREM 3.5. **(The spectral decomposition theorem)** *Let ϕ be an expansive flow on a compact TVS-cone metric space (X,d). If ϕ has the POTP, then its nonwandering set $\Omega(\phi)$ can be uniquely represented in the form $\Omega(\phi) = C_1 \cup \cdots \cup C_m$, where $C_1, \cdots, C_m$ are chain components.*

Proof. By Proposition 2.3, we obtain that $\Omega(\phi) = \mathrm{CR}(\phi)$. It is well known that $\mathrm{CR}(\phi)$ has a decompositon into chain components $\{C_i\}$.

By Lemma 2.2 and Lemma 3.3, chain components are open and closed in $\Omega(\phi)$, and are invariant. By Proposition 3.4, ϕ is topologically transitive on each component. Since $\Omega(\phi)$ is compact, $\Omega(\phi)$ is uniquely expressed as a finite disjoint union $\Omega(\phi) = \bigcup_{i=1}^{m} C_i$.

□

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